

Invertibility of infinitely divisible continuous-time moving average processes

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Abstract

This paper gives sufficient conditions for the invertibility of continuous time moving average processes of the form $X_t = \int_{\mathbb{R}} f(t-s) dL_s$, where f is a deterministic function and L a Lévy process. Such conditions are entirely determined by f and no moments of the Lévy process are required.

1 Introduction

The class of stationary processes plays an important role in time series analysis thanks to its tractability, flexibility, and good performance describing data from different scientific areas. Among these processes, the subclass of stationary infinitely divisible processes is of high interest. A stationary infinitely divisible process is a stationary process whose finite dimensional distributions are infinitely divisible. Thus, stationarity captures the autocorrelation structure observed in the data while the infinite divisibility incorporates to the model a class of well-behaved probability distributions. These probabilistic laws are able to reproduce many of the stylized properties present in empirical data such as fat tails and local Gaussianity (mixed Gaussian distributions).

In the context of time series, the concept of *invertibility* for discrete-time moving average processes refers to the question of whether the driving white noise can be recovered, in a pathwise sense, by the observed process. Further, it plays an important role for the characterization of *causality*, i.e. the current state of a given system is not influenced by its future states (for a precise definition see Definition 6 below). Invertibility and causality have been extensively studied in the square integrable case in which necessary and sufficient conditions have been established, see for instance Brockwell and Davis (1986). However, such characterization does not hold in the case when the white noise is not square integrable.

In this paper we study the invertibility property for the class of *continuous-time moving average processes driven by a Lévy process*. That is, the observed process $(X_t)_{t \in \mathbb{R}}$ admits the following spectral representation

$$X_t := \int_{\mathbb{R}} f(t-s) dL_s, \quad t \in \mathbb{R}, \quad (1)$$

where f is a measurable function, often called *kernel*, and L is a Lévy process.

Comte and Renault (1996) studied the case when L is a Brownian motion. Under certain regularities on the kernel, the authors gave necessary and sufficient conditions for the invertibility and causality of X . Their results include a subclass of Gaussian Volterra processes, i.e. X can be written as in (1), but the kernel f is not written as a shifted function but as a function of (t, s) instead. Cohen and Maejima (2011) considered the family of fractional Lévy processes driven by a centered and square integrable Lévy process, in other words, L has mean zero with finite second moment and

$f(t, s) = (t - s)_+^\alpha - (-s)_+^\alpha$. Here $\alpha \in (0, 1/2)$ and $(x)_+$ represents the positive part of $x \in \mathbb{R}$. By showing that the family $(f(t, \cdot))_{t \in \mathbb{R}}$ is dense in \mathcal{L}^2 , the authors established the invertibility of X . When L has mean zero and finite second moment, their argument can be used to show that processes of the form (1) are invertible whenever the family $(f(t - \cdot))_{t \in \mathbb{R}}$ is dense in \mathcal{L}^2 .

Recently, Brockwell and Lindner (2009) studied the invertibility and the causality of CARMA processes. They showed that the conditions for causality and invertibility are the same as in the discrete-time case. In the same context, Ferrazzano and Fuchs (2013) proposed an estimator for L_t based on a discrete-time approximation of X at time t . That estimator is consistent whenever L is centered and square integrable.

In this work, we will consider general processes of the form (1), i.e. no moments on the background driving Lévy processes are assumed. We give sufficient conditions on f for which X becomes invertible. Furthermore, we study a particular case where the causality holds. This case is not covered in the references mentioned above.

The present paper is organized as follows. Section 2 introduces the notation and some background on infinite divisibility, stochastic integration with respect to Lévy bases, and Orlicz spaces. In Section 3, we study the invertibility problem of infinitely divisible continuous-time moving average processes and we show some examples. Under the invertibility property, we show in Section 4 that the process (1) is causal when the kernel is proportional to a gamma density. Section 7 concludes.

2 Preliminaries and basic results

Throughout this paper $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$ denotes a filtered probability space satisfying the usual conditions of right-continuity and completeness. A two-sided \mathbb{R}^d -valued Lévy process $(L_t)_{t \in \mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a stochastic process taking values in \mathbb{R}^d with independent and stationary increments whose sample paths are almost surely càdlàg. We say that $(L_t)_{t \in \mathbb{R}}$ is an (\mathcal{F}_t) -Lévy process if for all $t > s$, $L_t - L_s$ is \mathcal{F}_t -measurable and independent of \mathcal{F}_s .

By $ID(\mathbb{R}^d)$ we mean the space of infinitely divisible distributions on \mathbb{R}^d . Any Lévy process is infinitely divisible and L_1 has a Lévy-Khintchine representation given by

$$\log \hat{\mu}(z) = i \langle z, \gamma \rangle - \frac{1}{2} \langle z, Bz \rangle + \int_{\mathbb{R}^d} \left[e^{i \langle z, x \rangle} - 1 - i \langle \tau(x), z \rangle \right] \nu(dx), \quad z \in \mathbb{R}^n,$$

where $\hat{\mu}$ is the characteristic function of the law of L_1 , $\gamma \in \mathbb{R}^d$, B is a symmetric nonnegative definite matrix on $\mathbb{R}^{d \times d}$, and ν is a Lévy measure, i.e. $\nu(\{0^d\}) = 0$, with 0^d denoting the origin in \mathbb{R}^d , and $\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(dx) < \infty$. Here, we assume that the truncation function τ is given by $\tau(x_1, \dots, x_n) = \left(\frac{x_i}{1 \vee |x_i|} \right)_{i=1}^n$, $(x_1, \dots, x_n) \in \mathbb{R}^n$.

An *infinitely divisible continuous-time moving average* (IDCMA) process is a stochastic process $(X_t)_{t \in \mathbb{R}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$ given by the following formula

$$X_t := \int_{\mathbb{R}} f(t - s) dL_s, \quad t \in \mathbb{R}, \quad (2)$$

where f is a deterministic function and L is a Lévy process with triplet (γ, B, ν) . Observe that X is strictly stationary and infinitely divisible in the sense of Barndorff-Nielsen et al. (2006) and Barndorff-Nielsen et al. (2015).

A *Lévy semistationary process* (\mathcal{LSS}) on \mathbb{R}^d is a stochastic process $(Y_t)_{t \in \mathbb{R}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$ which is described by the following dynamics

$$Y_t = \theta + \int_{-\infty}^t g(t - s) \sigma_s dL_s + \int_{-\infty}^t q(t - s) a_s ds, \quad t \in \mathbb{R}, \quad (3)$$

where $\theta \in \mathbb{R}^d$, L is a Lévy process with triplet (γ, B, ν) , g and q are deterministic functions such that $g(x) = q(x) = 0$ for $x \leq 0$, and σ and a are adapted càdlàg processes. When L is a two-sided Brownian motion Y is called a *Brownian semistationary process* (\mathcal{BSS}). In absence of drift and stochastic volatility, a \mathcal{LSS} is an IDCMA. For further references to theory and applications of Lévy semistationary processes, see for instance Veraart and Veraart (2014) and Benth et al. (2014). See also Brockwell et al. (2013).

2.1 Stochastic integration on the real line

In the following, we present a short review of Rajput and Rosiński (1989) and Sato (2006) concerning the existence of stochastic integrals of the form $\int_{\mathcal{S}} f(s) L(ds)$, where $f : \mathcal{S} \rightarrow \mathbb{R}$ is a measurable function and L a homogeneous Lévy basis. Later, we use it to explain how the integral in (2) is to be understood.

Let \mathcal{S} be a non-empty set and \mathcal{R} a δ -ring of subsets of \mathcal{S} having the property that there exists an increasing sequence $\{S_n\} \subset \mathcal{S}$ with $\bigcup_n S_n = \mathcal{S}$. An \mathbb{R}^d -valued stochastic field $L = \{L(A) : A \in \mathcal{R}\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called *independently scattered random measure* (i.s.r.m. for short), if for every sequence $\{A_n\}_{n \geq 1}$ of disjoint sets in \mathcal{R} , the random variables $(L(A_n))_{n \geq 1}$ are independent, and if $\bigcup_{n \geq 1} A_n$ belongs to \mathcal{R} , then we also have

$$L\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} L(A_n), \quad \text{a.s.},$$

where the series is assumed to converge almost surely. When the law of $L(A)$ belongs to $ID(\mathbb{R}^d)$ for any $A \in \mathcal{R}$, L is called a *Lévy basis*. Any Lévy basis admits a Lévy-Khintchine representation:

$$C\{z \sharp L(A)\} = \int_A \psi(z, s) c(ds), \quad z \in \mathbb{R}^d, A \in \mathcal{R},$$

where $C\{\theta \sharp X\}$ denotes the cumulant function of a random variable X and for any $z \in \mathbb{R}^d$ and $s \in \mathcal{S}$

$$\psi(z, s) := i\langle z, \gamma(s) \rangle - \frac{1}{2}\langle z, B_s z \rangle + \int_{\mathbb{R}^d} \left[e^{i\langle z, x \rangle} - 1 - i\langle \tau(x), z \rangle \right] \rho(s, dx). \quad (4)$$

The functions γ, B , and $\rho(\cdot, dx)$ are measurable with B being non-negative definitive and $\rho(s, \cdot)$ is a Lévy measure for every $s \in \mathcal{S}$. The measure c is defined on $\mathcal{B}_{\mathcal{S}} := \sigma(\mathcal{R})$ and is called the *control measure* of L . The quadruplet (γ, B, ρ, c) will be called *characteristic quadruplet* of L . When γ, B and $\rho(\cdot, dx)$ do not depend on s and c is the Lebesgue measure (up to a constant) L will be called *homogeneous*. In this case we say that L has characteristic triplet (γ, B, ρ) . When $\mathcal{S} = \mathbb{R}^k$, homogeneity is equivalent to the condition $L(A + x) \stackrel{d}{=} L(A)$ for any $x \in \mathbb{R}^k$.

If we put $\mathcal{R} = \mathcal{B}_b(\mathbb{R}^k)$, the bounded Borel sets, and add the extra condition $L(\{x\}) = 0$ a.s. for all $x \in \mathbb{R}^k$, L has a Lévy-Itô decomposition: we have that for a given $A \in \mathcal{R}$, almost surely

$$L(A) = \int_A \gamma(s) c(ds) + W(A) + \int_A \int_{|x| > 1} x N(dx ds) + \int_A \int_{|x| \leq 1} x \tilde{N}(dx ds),$$

where W is a centered Gaussian process with $\mathbb{E}(W(A)W'(A')) = \int_{A \cap A'} B_s c(ds)$ for all $A, A' \in \mathcal{R}$, \tilde{N} and N are compensated and non-compensated Poisson random measures on $\mathbb{R}^k \times \mathbb{R}$ with intensity $\rho(s, dx) c(ds)$, respectively. Additionally, W and N are independent. See Pedersen (2003) for more details.

In the homogeneous case, when $\mathcal{S} = \mathbb{R}$, the process characterized by $L_t - L_s := L((s, t])$ is an (\mathcal{F}_t) -Lévy process. Reciprocally, if $(L_t)_{t \in \mathbb{R}}$ is a Lévy process on \mathbb{R}^d , the random measure characterized by $L((s, t]) := L_t - L_s$ for $s \leq t$ is a homogeneous Lévy basis on \mathbb{R}^d . Therefore, in the present context, homogeneous Lévy bases and Lévy processes are in bijection.

By $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ we mean the space of \mathbb{R}^d -valued random variables endowed with the convergence in probability. Let ϑ be the space of simple functions on $(\mathcal{S}, \mathcal{R})$, i.e. $f \in \vartheta$ if and only if f can be written as

$$f(s) = \sum_{i=1}^k a_i \mathbf{1}_{A_i}(s), \quad s \in \mathcal{S},$$

where $A_i \in \mathcal{R}$ and $a_i \in \mathbb{R}$ for $i = 1, \dots, k$. For any $f \in \vartheta$, define the linear operator $m : \vartheta \rightarrow \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ by

$$m(f) := \sum_{i=1}^k a_i L(A_i).$$

Typically, the stochastic integral with respect to L is a linear extension of the operator of the form (4) to a suitable space, let us say $I_m(\gamma, B, \rho, c)$, such that $m(f)$ can be approximated by simple integrals of elements of ϑ . More precisely, if m can be extended to $I_m(\gamma, B, \rho, c)$ and ϑ is dense in this set, we say that f is L -integrable or $f \in I_m(\gamma, B, \rho, c)$ and we define its stochastic integral with respect to L as

$$\int_{\mathcal{S}} f(s) L(ds) := \mathbb{P}\text{-}\lim_{n \rightarrow \infty} m(f_n),$$

provided that $f_n \in \vartheta$ and $f_n \rightarrow f$ c -a.e.

In Rajput and Rosiński (1989) (see also Sato (2006)), it has been shown that the simple integral (4) can be extended to

$$I_m(\gamma, B, \rho, c) = \left\{ f : (\mathcal{S}, \mathcal{B}_{\mathcal{S}}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) : \int_{\mathcal{S}} \Phi(|f(s)|, s) c(ds) < \infty \right\}, \quad (5)$$

where

$$\Phi(r, s) := H(r, s) + tr(B_s)r^2 + \int_{\mathbb{R}^d} 1 \wedge |rx|^2 \rho(s, dx), \quad r \in \mathbb{R}, s \in \mathcal{S}, \quad (6)$$

with

$$H(r, s) := \left| \gamma(s)r + \int_{\mathbb{R}^d} [\tau(xr) - r\tau(x)] \rho(s, dx) \right|, \quad r \in \mathbb{R}, s \in \mathcal{S}.$$

When $f \in I_m(\gamma, B, \rho, c)$, $\int_{\mathcal{S}} f(s) L(ds)$ is infinitely divisible with characteristic triplet (γ^f, B^f, ν^f) given by

$$\begin{aligned} \gamma^f &= \int_{\mathbb{R}} \gamma(s) f(s) ds + \int_{\mathbb{R}} \int_{\mathbb{R}^d} [\tau(xf(s)) - f(s)\tau(x)] \rho(s, dx) ds, \\ B^f &= \int_{\mathbb{R}} f^2(s) B_s ds, \\ \nu^f(A) &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_A(f(s)x) \rho(s, dx) ds, \quad A \in \mathcal{B}(\mathbb{R}^d), \end{aligned}$$

and

$$C\{\theta \int_{\mathcal{S}} f(s) L(ds)\} = \int_{\mathcal{S}} \psi(f(s)\theta, s) c(ds), \quad \theta \in \mathbb{R}.$$

with ψ as in (4). Note that in the homogeneous case Φ does not depend on s .

Let $(L_t)_{t \in \mathbb{R}}$ be a Lévy process with characteristic triplet (γ, B, ν) and $L = \{L(A) : A \in \mathcal{B}_b(\mathbb{R})\}$ its associated homogeneous Lévy basis. We say that a measurable function f is integrable with respect to $(L_t)_{t \in \mathbb{R}}$, if $f \in I_m(\gamma, B, \nu, ds)$. In this case we write $f \in I_m(\gamma, B, \nu)$ and we put $\int_{\mathbb{R}} f(s) dL_s := \int_{\mathbb{R}} f(s) L(ds)$.

2.2 Orlicz spaces and stochastic integrals

In this part, we present some properties of Orlicz spaces that will be used in this paper. Also, we relate such spaces with the space of integrable functions with respect to a Lévy process. For a comprehensive introduction to Orlicz spaces, we refer to Rao and Ren (1994).

Let $(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \mu)$ be a σ -additive measure space and consider $\Psi : \mathbb{R} \rightarrow [0, \infty]$ to be a Young function, i.e. Ψ is even and convex with $\Psi(s) = 0$ if and only if $s = 0$. Denote

$$\mathcal{L}_{\Psi} := \left\{ f : (\mathcal{S}, \mathcal{B}_{\mathcal{S}}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) : \int_{\mathcal{S}} \Psi(|f(s)|) \mu(ds) < \infty \right\}.$$

Notice that, if $f \in \mathcal{L}_{\Psi}$, then by Jensen's inequality

$$\Psi\left(\frac{1}{\mu(S_n)} \int_{S_n} |f(s)| \mu(ds)\right) \leq \frac{1}{\mu(S_n)} \int_{S_n} \Psi(|f(s)|) \mu(ds) < \infty,$$

with $S_n \uparrow \mathcal{S}$ and $\mu(S_n) < \infty$. This means $\int_{S_n} |f(s)| \mu(ds) < \infty$, i.e. f is locally integrable.

If Ψ fulfills the Δ_2 -condition, that is $\Psi(2x) \leq K\Psi(x)$ for some $K > 0$ and any $x \geq 0$, then \mathcal{L}_{Ψ} is a linear space. Moreover, \mathcal{L}_{Ψ} endowed with the pseudo norm

$$\|f\|_0 := \inf \left\{ a > 0 : \int_{\mathbb{R}} \Psi(a^{-1}|f(s)|) ds \leq 1 \right\}, \quad (7)$$

is a separable Banach space. The pseudo norm $\|\cdot\|_0$ is called *Luxemburg norm*. It is possible to define an equivalent norm to $\|\cdot\|_0$ as the following proposition shows. For a proof see Rao and Ren (1994).

Proposition 1 *Assume that Ψ fulfills the Δ_2 -condition. Then, the space \mathcal{L}_{Ψ} endowed with the norm*

$$\|f\|_{\Psi} := \sup \left\{ \int_{\mathcal{S}} |f(s)g(s)| \mu(ds) : \int_{\mathcal{S}} \overline{\Psi}(|g(s)|) \mu(ds) \leq 1 \right\},$$

is a separable Banach space. Here $\overline{\Psi}$ denotes the complementary function of Ψ (which is also a Young function), i.e.

$$\overline{\Psi}(x) := \sup_{y \geq 0} \{ |x|y - \Psi(y) \}, \quad x \in \mathbb{R}. \quad (8)$$

Moreover, we have that for any $f \in \mathcal{L}_{\Psi}$

$$\|f\|_0 \leq \|f\|_{\Psi} \leq 2\|f\|_0. \quad (9)$$

The norm $\|\cdot\|_{\Psi}$ is called *Orlicz norm*. Therefore, we can infer properties of $(\mathcal{L}_{\Psi}, \|\cdot\|_0)$ through $(\mathcal{L}_{\Psi}, \|\cdot\|_{\Psi})$ and vice versa. The next theorem describes \mathcal{L}'_{Ψ} , the dual of \mathcal{L}_{Ψ} , and it will play an important role for the proof of our main result in the next section. As before, we remit its proof to (Rao and Ren, 1994, Theorem 6, page 105.).

Theorem 2 *Suppose that $(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \mu)$ is a σ -additive measure space and Ψ is a Young function fulfilling the Δ_2 -condition. Then \mathcal{L}'_{Ψ} is isometrically isomorphic to $\mathcal{L}_{\overline{\Psi}}$, where $\overline{\Psi}$ is as in (8). More precisely, for any $T \in \mathcal{L}'_{\Psi}$ there exists a unique $g \in \mathcal{L}_{\overline{\Psi}}$, such that*

$$T(f) = \int_{\mathcal{S}} f(s)g(s) \mu(ds), \quad f \in \mathcal{L}_{\Psi},$$

and $\|T\| := \sup \{ |T(f)| : \|f\|_0 \leq 1 \} = \|g\|_{\overline{\Psi}}$.

Recall that in a Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, a collection $F = (f_{\alpha})_{\alpha \in \Lambda}$ is said to be dense if $\overline{F} = \mathcal{X}$ under the norm $\|\cdot\|_{\mathcal{X}}$. From the previous theorem, Proposition 1 and the Hahn-Banach Theorem we get:

Corollary 3 *A collection $F = (f_\alpha)_{\alpha \in \Lambda} \subset \mathcal{L}_\Psi$ is dense in \mathcal{L}_Ψ under $\|\cdot\|_\Psi$ (equivalently under $\|\cdot\|_0$) if and only if*

$$T(f_\alpha) = \int_{\mathcal{S}} f_\alpha(s) g(s) \mu(ds) = 0, \quad \forall \alpha \in \Lambda,$$

with $g \in \mathcal{L}_{\overline{\Psi}}$, implies that $g \equiv 0$ μ -almost everywhere.

Fix a Lévy process $(L_t)_{t \in \mathbb{R}}$ with characteristic triplet (γ, B, ν) . The space of L integrable functions is intimately related with an Orlicz space induced by the triplet (γ, B, ν) , as the following reasoning shows: Let

$$\Psi(r) := \sup_{c \leq 1} H(cr) + tr(B)r^2 + \int_{\mathbb{R}^d} 1 \wedge |rx|^2 \nu(dx), \quad r \in \mathbb{R}, \quad (10)$$

where

$$H(r) := \left| \gamma r + \int_{\mathbb{R}^d} [\tau(xr) - r\tau(x)] \nu(dx) \right|, \quad r \in \mathbb{R}.$$

Due to Lemma 3.1 in Rajput and Rosiński (1989), Ψ is a Young function satisfying the Δ_2 -condition. Thus, thanks to (5) and Lemma 2.8 in Rajput and Rosiński (1989) the Orlicz space $(\mathcal{L}_\Psi, \|\cdot\|_0)$ with μ being the Lebesgue measure, coincides with $I_m(\gamma, B, \nu)$. Observe that from (9), $I_m(\gamma, B, \nu) = (\mathcal{L}_\Psi, \|\cdot\|_\Psi)$. Hence, in what follows, for a given Lévy process with triplet (γ, B, ν) , $(\mathcal{L}_\Psi, \|\cdot\|_\Psi)$ (equivalently $(\mathcal{L}_\Psi, \|\cdot\|_0)$) will represent the Orlicz space induced by the Young function defined in (10).

Remark 4 Although the Lévy processes under consideration are \mathbb{R}^d -valued, the space $(\mathcal{L}_\Psi, \|\cdot\|_\Psi)$ contains only real-valued functions.

The following properties of the mapping associated to $I_m(\gamma, B, \nu)$ will be useful for the rest of the paper, see Rajput and Rosiński (1989) for a proof:

Theorem 5 *Let $(L_t)_{t \in \mathbb{R}}$ be a Lévy process with triplet (γ, B, ν) and $(\mathcal{L}_\Psi, \|\cdot\|_0)$ its associated Orlicz space. Then*

1. The mapping $(f \in \mathcal{L}_\Psi) \mapsto (\int_{\mathcal{S}} f(s) dL_s \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}))$ is continuous in $(\mathcal{L}_\Psi, \|\cdot\|_0)$, i.e. if $f_n \rightarrow 0$ in $(\mathcal{L}_\Psi, \|\cdot\|_0)$, then $\int_{\mathcal{S}} f_n(s) L(ds) \rightarrow 0$ in $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$;
2. If L is symmetric, then $(f \in \mathcal{L}_\Psi) \mapsto (\int_{\mathcal{S}} f(s) L(ds) \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}))$ is an isomorphism between $(\mathcal{L}_\Psi, \|\cdot\|_0)$ and $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$. The same holds when L is centered.

When L has finite p -moment for $p > 0$, it is possible to consider a subspace of \mathcal{L}_Ψ which can be mapped into a subspace of $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$. Indeed, for every $r \in \mathbb{R}$, let

$$\Psi_p(r) := \sup_{c \leq 1} H(cr) + tr(B)r^2 + \int_{\mathbb{R}} \left[|xr|^p 1_{\{|xr| > 1\}} + |xr|^2 1_{\{|xr| \leq 1\}} \right] \nu(dx).$$

Then Ψ_p is once again a Young function satisfying the Δ_2 -condition and $\Psi_0 = \Psi$. In this case, the space $(\mathcal{L}_{\Psi_p}, \|\cdot\|_0)$ is the space of integrable functions for which the stochastic integral has finite p -moment. Furthermore, Theorem 5 still holds, i.e. the mapping $(f \in \mathcal{L}_{\Psi_p}) \mapsto (\int_{\mathcal{S}} f(s) dL_s \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}))$ is continuous in $(\mathcal{L}_{\Psi_p}, \|\cdot\|_0)$ (equivalently in $(\mathcal{L}_\Psi, \|\cdot\|_\Psi)$) and it is an isomorphism when L is symmetric or centered. Let us remark that $(\mathcal{L}_{\Psi_p}, \|\cdot\|_0)$ is isometric (continuously embedded) to some Hilbert space if and only if L is a centered square integrable martingale, i.e. $p = 2$. In this particular situation (square integrable martingale) $\mathcal{L}_{\Psi_2} = \mathcal{L}^2(\mathbb{R}, lds)$ for $l > 0$ and it is isometric to $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. For a proof of this fact see Kaminska (1997).

3 Invertibility of IDCMA processes

In this section we present the main result of this paper. We show that given an IDCMA process it is always possible to recover the background Lévy process by imposing some conditions directly on the kernel. We would like to emphasize that no finite moments in the background driving Lévy process are required. Moreover, in the square integrable case we show that such conditions are necessary as well.

Let us start by recalling the notions of causality and non-causality. Let $(X_t)_{t \in \mathbb{Z}}$ be a discrete-time weak stationary process admitting the following Wold representation

$$X_t = \sum_{j \in \mathbb{Z}} \theta_j \varepsilon_{t-j} = \Theta(B) \varepsilon_t, \quad t \in \mathbb{Z},$$

where the process $(\varepsilon_t)_{t \in \mathbb{Z}}$ is a mean zero weak stationary white noise, $\sum_{j \in \mathbb{Z}} |\theta_j| < \infty$, B is the lag operator and

$$\Theta(z) = \sum_{j \in \mathbb{Z}} \psi_j z^j, \quad z \in \mathbb{C}, |z| < 1.$$

Observe that if Θ is invertible, then almost surely

$$\varepsilon_t = \Theta^{-1}(B) X_t = \sum_{j \in \mathbb{Z}} \pi_j X_{t-j}, \quad t \in \mathbb{Z}.$$

Thus, $\varepsilon_t \in \overline{\text{span}}\{X_s\}_{s \in \mathbb{Z}}$ for any $t \in \mathbb{Z}$, where the closure is taken in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. In the discrete-time case this property is called invertibility. See for instance Brockwell and Davis (1986). A necessary and sufficient condition for invertibility is that the lag polynomial Θ does not vanish inside the unitary circle, this is, $\Theta(z) \neq 0$ for all $|z| < 1$ or, equivalently, for all $|\omega| < \pi$

$$0 \neq \sum_{j \in \mathbb{Z}} \theta_j e^{-ij\omega} = \Theta(e^{-i\omega}) =: \hat{\Theta}(\omega).$$

Note that $\hat{\Theta}$ is the discrete Fourier transform of $(\theta_j)_{j \in \mathbb{Z}}$. Therefore, in the discrete-time case a necessary and sufficient condition for invertibility is that the Fourier transform of the weights does not vanish and the process X is square integrable (equivalently for ε).

Observe that invertibility itself does not tell us anything about adaptability of the process. For instance, if X follows an autoregressive dynamics, i.e.

$$X_t = \theta X_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z},$$

then X has stationary a solution if and only if $\theta \neq 1$. In particular, if $|\theta| < 1$

$$X_t = \sum_{j \geq 0} \theta^j \varepsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{11}$$

and if $|\theta| > 1$

$$X_t = - \sum_{j \geq 0} \theta^{-j} \varepsilon_{t+j}, \quad t \in \mathbb{Z}. \tag{12}$$

Note that in (11) X only depends on the past innovations of ε contrary to that in (12), in which X is expressed in terms of the future innovations of ε . When X admits a representation as in (11), it is called causal and for the case of (12) it is called non-causal. However, it is obvious that ε only depends on the past innovations of X , i.e. ε admits a causal representation. This property is usually called *invertibility in the causal sense*.

In analogy with the discrete-time framework, we introduce the notion of invertibility for an IDCMA.

Definition 6 Let X be as in (2). We say that X is causal if f has support on $[0, \infty)$. In the same context, we are going to say that X is invertible if $L_t \in \overline{\text{span}}\{X_s\}_{s \in \mathbb{R}}$ for any $t \in \mathbb{R}$, where the closure is in $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, we call X an invertible process in the causal sense if for a given $t \in \mathbb{R}$ we have that $L_t \in \overline{\text{span}}\{X_s\}_{s \leq t}$.

A natural question appears, as in the discrete-time case, is $\widehat{f} \neq 0$ a sufficient (necessary) condition for the invertibility of an IDCMA? For the rest of the section we will consider this question.

We would like to point out that necessary and sufficient conditions for the invertibility of Gaussian stationary process in continuous-time have been given in Comte and Renault (1996). Here the authors consider an Itô semimartingale which admits a so-called Volterra representation. However, in general an IDCMA process is not a semimartingale (e.g. $f(s) = e^{-s} s^\alpha \mathbf{1}_{\{s > 0\}}$ with $-1 < \alpha < 1/2$) and the noise does not need to be Gaussian.

By the discussion above, the invertibility in the discrete-time case is given in the $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ sense, which requires moments in the noise. In this paper we are not going to assume neither the existence of moments in the background Lévy process nor the semimartingale property.

Now, we present our main result:

Theorem 7 Let $(L_t)_{t \in \mathbb{R}}$ be a Lévy process with characteristic triplet (γ, B, ν) and $f \in \mathcal{L}_\Psi$. Then X as in (2) is well defined. Moreover, if $f \in \mathcal{L}_\Psi \cap \mathcal{L}^1(dx)$ with non-vanishing Fourier transform we have that

$$\overline{\text{span}}\{X_t\}_{t \in \mathbb{R}} = \overline{\text{span}}\{L_t\}_{t \in \mathbb{R}}, \quad \text{in } \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}). \quad (13)$$

Remark 8 In a very informal way, Theorem 7 says that there exists a measurable function g such that $L_t = \int_{\mathbb{R}} g(t, s) dX_s$ for all $t \in \mathbb{R}$. However, since X is not in general a semimartingale, such integral may not be well defined.

Before presenting the proof of this theorem, we are going to discuss several important examples.

Example 9 (Ornstein-Uhlenbeck processes) Suppose that L is a Lévy process with characteristic triplet (γ, B, ν) . Let

$$f(s) := e^{-\lambda s} \mathbf{1}_{\{s \geq 0\}}, \quad s \in \mathbb{R}.$$

Then X , the resulting IDCMA process, is the classic OU process driven by L . It is well known that $f \in \mathcal{L}_\Psi$ if and only if $\int_{|x| > 1} \log(|x|) \nu(dx) < \infty$. Moreover, since \widehat{f} , the Fourier transform of f , never vanishes, we conclude that an OU process is invertible. Furthermore, due to the Langevin equation, we can show that such IDCMA is in fact invertible in the causal sense. Indeed, since almost surely

$$L_t - L_s = X_t - X_s + \lambda \int_s^t X_u du, \quad t \geq s,$$

then $L_t \in \overline{\text{span}}\{X_s\}_{s \leq t}$, as claimed. This is an example of a causal IDCMA which is invertible in the causal sense.

Example 10 (LSS with a Gamma kernel) Take L to be a Lévy process with characteristic triplet (γ, B, ν) . Let $\alpha > -1$ and consider

$$f(s) = \varphi_\alpha(s) := e^{-s} s^\alpha \mathbf{1}_{\{s > 0\}}, \quad s \in \mathbb{R}. \quad (14)$$

It has been shown in Basse-O'Connor (2013), c.f. Pedersen and Sauri (2015), that $\varphi_\alpha \in \mathcal{L}_\Psi$ if and only if the following two conditions are satisfied:

1. $\int_{|x| > 1} \log(|x|) \nu(dx) < \infty$,
2. One of the following conditions holds:

- (a) $\alpha > -1/2$;
- (b) $\alpha = -1/2$, $B = 0$ and $\int_{|x| \leq 1} |x|^2 |\log(|x|)| \nu(dx) < \infty$;
- (c) $\alpha \in (-1, -1/2)$, $B = 0$ and $\int_{|x| \leq 1} |x|^{-1/\alpha} \nu(dx) < \infty$.

In this case X , the associated IDCMA process, is called a *Lévy semistationary process with a gamma kernel*. See Pedersen and Sauri (2015) for more properties on this process. Note that the Fourier transform of φ_α is given by

$$\hat{\varphi}_\alpha(\xi) = \frac{\Gamma(\alpha+1)}{\sqrt{2\pi}} (1-i\xi)^{-\alpha-1}, \quad \xi \in \mathbb{R}.$$

Hence, X is invertible. In the next section we will see that X is not only invertible but invertible in the causal sense.

Example 11 ($CARMA(p, q)$) The continuous auto-regressive moving average process with parameters $p > q$ ($CARMA(p, q)$) is the stationary process given by $X_t = \mathbf{b}'Y_t$ where Y follows the following SDE

$$dY_t = AY_t dt + e_p dL_t,$$

where L is a real-valued Lévy process with characteristic triplet (γ, B, ν) , $\mathbf{b} = (b_0, \dots, b_{p-1})'$, $\mathbf{e}_p = (0, 0, \dots, 1)'$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & a_{p-1} & a_{p-2} & \cdots & -a_1 \end{bmatrix}.$$

where $a_1, \dots, a_p, b_0, \dots, b_{p-1}$ are complex numbers such that $b_q \neq 0$ and $b_j = 0$ for $j > q$. It has been shown in Brockwell and Lindner (2009) that X admits the representation

$$X_t = \int_{\mathbb{R}} g(t-s) dL_s, \quad t \in \mathbb{R},$$

with

$$g(s) = b'e^{As}\mathbf{e}_p \mathbf{1}_{\{s>0\}},$$

provided that $\int_{|x|>1} \log(|x|) \nu(dx) < \infty$ and the roots of the polynomial $a(\lambda) = a_p + a_{p-1}\lambda + \dots + a_1\lambda^{p-1} + \lambda^p$, $\lambda \in \mathbb{C}$, have strictly negative real part. This condition is to guarantee that A is negative definite. We observe that X is a causal IDCMA.

Since in this case

$$\hat{g}(\xi) = \frac{b(-i\xi)}{a(-i\xi)}, \quad \xi \in \mathbb{R},$$

with $b(\lambda) = b_0 + b_1\lambda + \dots + b_{p-1}\lambda^{p-1}$, $\lambda \in \mathbb{C}$, we conclude that a $CARMA(p, q)$ is invertible if the roots of the polynomial b have non-vanishing real part, i.e. if $b(\lambda^*) = 0$ then $\text{Re}\lambda^* \neq 0$, and a and b have no common roots. Observe that this condition coincides with the Assumption 1 in Ferrazzano and Fuchs (2013). Finally, since X is just a linear combination of real valued OU processes, from Example 9 we deduce that X is invertible in the causal sense.

Example 12 (Anticipating OU process) Consider the non-causal IDCMA

$$X_t := \int_t^\infty e^{-(s-t)} dL_s, \quad t \in \mathbb{R}.$$

Obviously X is neither $(\mathcal{F}_t)_{t \in \mathbb{R}}$ -adapted nor causal. However, it is easy to check that X fulfills a sort of Langevin equation, that is, almost surely,

$$\int_0^t X_s ds = L_t + X_t - X_0, \quad t \geq 0.$$

From this, we deduce that X is invertible in the causal sense.

For the proof of Theorem 7 we need some auxiliary lemmas.

Lemma 13 *Let $(L_t)_{t \in \mathbb{R}}$ be a Lévy process with characteristic triplet (γ, B, ν) and $(\mathcal{L}_\Psi, \|\cdot\|_0)$ its associated Orlicz space. Consider $(f_\alpha)_{\alpha \in \Lambda} \subset \mathcal{L}_\Psi$. If $\mathbf{1}_{(s,t]} \in \overline{\text{span}}(f_\alpha)_{\alpha \in \Lambda}$ under $\|\cdot\|_\Psi$ for $s \leq t$, then $L_t - L_s \in \overline{\text{span}}\left(\int_{\mathbb{R}} f_\alpha(s) dL_s\right)_{\alpha \in \Lambda}$ in $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. Let us start by noting that from (9), $g \in \overline{\text{span}}(f_\alpha)_{\alpha \in \Lambda}$ in $(\mathcal{L}_\Psi, \|\cdot\|_\Psi)$ if and only if $g \in \overline{\text{span}}(f_\alpha)_{\alpha \in \Lambda}$ in $(\mathcal{L}_\Psi, \|\cdot\|_0)$. Thus, if $\mathbf{1}_{(s,t]} \in \overline{\text{span}}(f_\alpha)_{\alpha \in \Lambda}$ under $\|\cdot\|_\Psi$ for $s \leq t$, then there exist $\theta^n := (\theta_i^n)_{i=1}^n \in \mathbb{R}^n$ and $\alpha^n := (\alpha_i^n)_{i=1}^n \subset \Lambda^n$ with $n \in \mathbb{N}$, such that $\|\langle \theta^n, F_{\alpha^n} \rangle - \mathbf{1}_{(s,t]}\|_0 \rightarrow 0$, where $F_{\alpha^n} := (f_\alpha)_{\alpha \in \alpha^n}$. Therefore, from Theorem 5, $\int_{\mathbb{R}} \langle \theta^n, F_{\alpha^n}(s) \rangle dL_s \xrightarrow{\mathbb{P}} L_t - L_s$, but since $\int_{\mathbb{R}} \langle \theta^n, F_{\alpha^n}(s) \rangle dL_s \in \text{span}\left(\int_{\mathbb{R}} f_\alpha(s) dL_s\right)_{\alpha \in \Lambda}$, it follows that $L_t - L_s \in \overline{\text{span}}\left(\int_{\mathbb{R}} f_\alpha(s) dL_s\right)_{\alpha \in \Lambda}$ in $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, as required. ■

Theorem 14 *Suppose that $f \in \mathcal{L}_\Psi \cap \mathcal{L}^1(dx)$ with non-vanishing Fourier transform. Then*

$$\overline{\text{span}}\{f(t - \cdot)\}_{t \in \mathbb{R}} = \mathcal{L}_\Psi. \quad (15)$$

We postpone the proof of this theorem. Now we proceed to prove the main result of this paper.

Proof of Theorem 7. Obviously $\overline{\text{span}}\{X_t\}_{t \in \mathbb{R}} \subset \overline{\text{span}}\{L_t\}_{t \in \mathbb{R}}$. From this and Lemma 13, we only need to check that for every $u > s$, the indicator function $\mathbf{1}_{(s,u]}$ belongs to $\overline{\text{span}}\{f(t - \cdot)\}_{t \in \mathbb{R}}$. Under our assumptions, Theorem 14 holds, thus $\overline{\text{span}}\{f(t - \cdot)\}_{t \in \mathbb{R}} = \mathcal{L}_\Psi$. In view that $\mathbf{1}_{(s,u]} \in \mathcal{L}_\Psi$ for any $u > s$, we deduce that $\mathbf{1}_{(s,u]} \in \overline{\text{span}}\{f(t - \cdot)\}_{t \in \mathbb{R}}$, which is enough. ■

Finally, we show Theorem 14. Recall that any function g in an Orlicz space (or its dual) is locally integrable.

Proof of Theorem 14. We are going to proceed as in Thuong (2000). Suppose the opposite, that is $\text{span}\{f(t - \cdot)\}_{t \in \mathbb{R}}$ is not dense in \mathcal{L}_Ψ . Then, from Corollary 3, there exists $g \in \mathcal{L}_{\Psi^*}$ not identically zero, such that

$$\int_{\mathbb{R}} f(t - s) g(s) ds = 0, \quad \text{for all } t \in \mathbb{R}. \quad (16)$$

This implies that $f * g = 0$, where $*$ denotes the convolution between f and g . Since $g \in \mathcal{L}_{\Psi^*}$, g is locally integrable, thus g induces a distribution. Moreover, by hypothesis $f \in \mathcal{L}^1(dx)$, therefore the Fourier transform $f * g$ can be defined in the distributional sense and equals $\widehat{f\widehat{g}}$, where \widehat{g} is the Fourier transform of the distribution induced by g . From (16) we get that $\widehat{f\widehat{g}} = 0$, but since \widehat{f} never vanishes, we deduce immediately that $\widehat{g} \equiv 0$. This implies necessarily that $g = 0$, which is a contradiction. Hence, $\text{span}\{f(t - \cdot)\}_{t \in \mathbb{R}}$ is dense in \mathcal{L}_Ψ or, in other words, equation (15) holds. ■

Remark 15 Observe that the reasoning in the previous proof holds for any function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, having non-vanishing Fourier transform. Therefore, Theorem 7 is also true for multiparameter processes of the form (2).

Corollary 16 *Let $(L_t)_{t \in \mathbb{R}}$ be a centered square integrable Lévy process. Then, X as in (2) is well defined if and only if $f \in \mathcal{L}^2(\mathbb{R}, ds)$. In addition, the Fourier transform of f does not vanish if and only if*

$$\overline{\text{span}}\{X_t\}_{t \in \mathbb{R}} = \overline{\text{span}}\{L_t\}_{t \in \mathbb{R}}, \quad \text{in } \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Proof. This follows immediately from the fact that in this case $(\mathcal{L}_{\Psi_p}, \|\cdot\|_0) = \mathcal{L}^2(\mathbb{R}, lds)$ and the mapping $(f \in \mathcal{L}^2(\mathbb{R}, Kds)) \mapsto (\int_{\mathcal{S}} f(s) dL_s \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}))$ is a continuous invertible isometry. ■

4 Invertibility in the causal sense of a \mathcal{LSS} with a gamma kernel

Using Theorem 7 and a stochastic Fubini Theorem, we show that a \mathcal{LSS} with a gamma kernel is invertible in the causal sense.

Let X be an IDCMA as in (2). In this case, it has been shown in Barndorff-Nielsen and Basse-O'Connor (2011) that X always admits a measurable modification. Furthermore, the following Stochastic Fubini Theorem for Lévy bases provides sufficient conditions for X to be integrable over time.

Theorem 17 (Fubini (Barndorff-Nielsen and Basse-O'Connor (2011))) *Let L be a centered Lévy basis with characteristic quadruplet $(\gamma(s), b(s), \rho(s, dx), c(ds))$ and $(U, \mathcal{B}(U), \mu)$ be a σ -finite measure space. Consider $f : U \times \mathcal{S} \rightarrow \mathbb{R}$ a $\mathcal{B}(U) \otimes \mathcal{B}_{\mathcal{S}}$ -measurable function such that $f(u, \cdot) \in \mathcal{L}_{\Psi}$ for all $u \in U$. Assume that for $A \in \mathcal{B}(U)$*

$$\int_A \|f(u, \cdot)\|_0 \mu(du) < \infty, \quad (17)$$

where $\|\cdot\|_0$ is as in (7). Then $f(\cdot, s) \in \mathcal{L}^1(U, \mathcal{B}(U), \mu)$ for c -almost every $s \in \mathcal{S}$ and the mapping $s \mapsto \int_A f(u, s) \mu(du)$ belongs to \mathcal{L}_{Ψ_1} . In this case, all the integrals below exist and almost surely

$$\int_A \left[\int_{\mathcal{S}} f(u, s) L(ds) \right] \mu(du) = \int_{\mathcal{S}} \left[\int_E f(u, s) \mu(du) \right] L(ds).$$

Moreover, if μ is finite, (17) is equivalent to

$$\int_A \int_{\mathcal{S}} \left[f^2(u, s) b^2(s) + \int_{\mathbb{R}} |xf(u, s)| \wedge |xf(u, s)|^2 c(ds) \right] \mu(du) < \infty. \quad (18)$$

Remark 18 Note that when L is homogeneous and $f(u, s) = g(u - s)$, with $g \in \mathcal{L}_{\Psi}$, (17) holds if and only if $\mu(A) < \infty$ and $g \in \mathcal{L}_{\Psi_1}$. Indeed, this follows from the fact that in this case

$$\|f(u, \cdot)\|_0 = \|g\|_0, \quad \text{for all } u \in U. \quad (19)$$

Using the previous theorem we can show the following:

Theorem 19 *Let $(L_t)_{t \in \mathbb{R}}$ be a centered Lévy process with characteristic triplet (γ, B, ν) and $(\mathcal{L}_{\Psi_1}, \|\cdot\|_0)$ be its associated Orlicz space. Consider*

$$X_t := \int_{-\infty}^t \varphi_{\alpha}(t - s) dL_s, \quad t \in \mathbb{R}, \quad (20)$$

with φ_{α} as in (14). Then $\varphi_{\alpha} \in \mathcal{L}_{\Psi_1}$ if and only if $\varphi_{\alpha} \in \mathcal{L}_{\Psi}$. Moreover, for any $-1 < \alpha < 0$, almost surely

$$\int_0^{\infty} X_{t-u} \mu(du) = k_{\alpha} \int_{-\infty}^t e^{-(t-s)} dL_s, \quad \text{for any } t \in \mathbb{R}, \quad (21)$$

with $k_{\alpha} > 0$ and $\mu(du) := \varphi_{-\alpha-1}(u) \mathbf{1}_{\{u \geq 0\}} du$.

Proof. Let us start by showing that $\varphi_\alpha \in \mathcal{L}_\Psi$ if and only if $\varphi_\alpha \in \mathcal{L}_{\Psi_1}$. Obviously if $\varphi_\alpha \in \mathcal{L}_{\Psi_1}$ we have that $\varphi_\alpha \in \mathcal{L}_\Psi$, so suppose that $\varphi_\alpha \in \mathcal{L}_\Psi$. Then conditions 1.-2. in Example 10 are satisfied. Now, since $|\varphi_\alpha| \leq c_1 \phi_\alpha$, where $c_1 > 0$ and

$$\phi_\alpha(s) := \begin{cases} s^\alpha \mathbf{1}_{\{0 < s \leq 1\}} + e^{-s} \mathbf{1}_{\{s > 1\}} & \text{for } -1 < \alpha < 0; \\ e^{-s} \mathbf{1}_{\{s \geq 0\}} & \text{for } \alpha \geq 0, \end{cases}$$

we only need to check that in this case $\phi_\alpha \in \mathcal{L}_{\Psi_1}$. If $\alpha > 0$, $\phi_\alpha \in \mathcal{L}_{\Psi_1}$ if and only if

$$\int_{\mathbb{R}} \int_0^\infty \left[|xe^{-s}| \wedge |xe^{-s}|^2 \right] ds \nu(dx) < \infty.$$

This follows by noting that

$$\begin{aligned} \int_{\mathbb{R}} \int_0^\infty \left[|xe^{-s}| \wedge |xe^{-s}|^2 \right] ds \nu(dx) &= \frac{1}{2} \int_{|x| \leq 1} |x|^2 \nu(dx) \\ &\quad + \int_{|x| > 1} \int_0^{\log(|x|)} |xe^{-s}| ds \nu(dx) \\ &\quad + \int_{|x| > 1} \int_{\log(|x|)}^\infty |xe^{-s}|^2 ds \nu(dx) \\ &= \frac{1}{2} \int_{\mathbb{R}} 1 \wedge |x|^2 \nu(dx) \\ &\quad + \int_{|x| > 1} \left[\frac{|x|^2 - 1}{|x|} \right] \nu(dx) < \infty, \end{aligned}$$

where we have used that $\int_{|x| > 1} |x| \nu(dx) < \infty$ (because L has first moment). Assume now that $\alpha \in (-1, 0)$. Once again, using 1.-2. in Example 10, we deduce that $\text{tr} B \int_0^\infty \phi_\alpha^2(s) ds < \infty$. Therefore, to show that $\phi_\alpha \in \mathcal{L}_{\Psi_1}$, it is enough to verify that $\int_{\mathbb{R}} \int_0^\infty \left[|x\phi_\alpha(s)| \wedge |x\phi_\alpha(s)|^2 \right] ds \nu(dx) < \infty$. From the calculations above, we see that

$$\int_{\mathbb{R}} \int_1^\infty \left[|x\phi_\alpha(s)| \wedge |x\phi_\alpha(s)|^2 \right] ds \nu(dx) \leq \int_{\mathbb{R}} \int_0^\infty \left[|xe^{-s}| \wedge |xe^{-s}|^2 \right] ds \nu(dx) < \infty.$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}} \int_0^1 \left[|x\phi_\alpha(s)| \wedge |x\phi_\alpha(s)|^2 \right] ds \nu(dx) &= \frac{1}{\alpha + 1} \int_{|x| > 1} |x| \nu(dx) \\ &\quad + \int_{|x| \leq 1} \int_0^{|x|^{-\frac{1}{\alpha}}} |xs^\alpha| ds \nu(dx) \\ &\quad + \int_{|x| \leq 1} \int_{|x|^{-\frac{1}{\alpha}}}^1 |xs^\alpha|^2 ds \nu(dx) \\ &= \frac{1}{\alpha + 1} \int_{\mathbb{R}} |x|^{-\frac{1}{\alpha}} \wedge |x| \nu(dx) \\ &\quad + \frac{1}{2\alpha + 1} \int_{|x| \leq 1} \left(|x|^2 - |x|^{-\frac{1}{\alpha}} \right) \nu(dx), \end{aligned}$$

which is finite due to conditions 1. and 2. in Example 10. Thus $\phi_\alpha \in \mathcal{L}_{\Psi_1}$.

All the above jointly with Theorem 17 imply that for any $\alpha > -1$ and $\beta > -1$ the following integrated

process

$$\begin{aligned} X_t^\mu &:= \int_0^\infty X_{t-u} \mu(du) \\ &= \int_{-\infty}^t \varphi_\beta(t-u) X_u du, \quad t \in \mathbb{R}, \end{aligned}$$

is well defined. In particular for $-1 < \alpha < 0$, $\beta = -\alpha - 1$ and every $t \in \mathbb{R}$, almost surely

$$\begin{aligned} X_t^\mu &= \int_{-\infty}^t e^{-(t-s)} \int_s^t (t-u)^{-\alpha-1} (u-s)^\alpha du dL_s \\ &= k_\alpha \int_{-\infty}^t e^{-(t-s)} dL_s, \end{aligned}$$

where $k_\alpha := \int_0^1 x^\alpha (1-x)^{-\alpha-1} dx < \infty$. ■

Remark 20 Observe that (21) was already shown in Barndorff-Nielsen et al. (2013) in the case when $(L_t)_{t \in \mathbb{R}}$ is a subordinator and $-1/2 < \alpha < 0$. Furthermore, we would like to emphasize that for $\alpha > 0$, (21) does not hold anymore. Indeed, suppose that $(L_t)_{t \in \mathbb{R}}$ is a subordinator and $\alpha > 0$. In this situation, the stochastic integral in (20) coincides with the Lebesgue-Stieltjes integral. Consequently, thanks to Tonelli's Theorem, for almost all $\omega \in \Omega$

$$\int_0^\infty X_{t-u}(\omega) \mu(du) = k_\alpha \int_{-\infty}^t e^{-(t-s)} dL_s(\omega), \quad t \in \mathbb{R},$$

where we have used that the mapping $(s, u) \mapsto \varphi_{-\alpha-1}(t-u) \varphi_\alpha(u-s)$ is measurable and non-negative for any $t \in \mathbb{R}$. By noting that in this case we have that $\int_0^1 x^\alpha (1-x)^{-\alpha-1} dx = +\infty$, we deduce immediately that $\int_0^\infty X_{t-u} \mu(du) = +\infty$, almost surely, or in other words (20) cannot hold in this case.

Corollary 21 *Let $(L_t)_{t \in \mathbb{R}}$ be a centered Lévy process with characteristic triplet (γ, B, ν) and $(\mathcal{L}_{\Psi_1}, \|\cdot\|_0)$ its associated Orlicz space. Consider*

$$X_t := \int_{-\infty}^t \varphi_\alpha(t-s) dL_s, \quad t \in \mathbb{R},$$

with φ_α as in (14). Then $L_t \in \overline{\text{span}}\{X_s\}_{s \leq t}$ for any $t \in \mathbb{R}$.

Proof. Let Z be the OU process induced by L for $\lambda = 1$. From the previous theorem, we have that $\overline{\text{span}}\{Z_s\}_{s \leq t} \subset \overline{\text{span}}\{X_s\}_{s \leq t}$, but from Example 9, we have that $L_t \in \overline{\text{span}}\{Z_s\}_{s \leq t}$ for all $t \in \mathbb{R}$. Hence, $L_t \in \overline{\text{span}}\{X_s\}_{s \leq t}$ for any $t \in \mathbb{R}$, from which the result follows. ■

5 Conclusions

This paper studies the invertibility of continuous-time moving averages processes driven by a Lévy process. We show that the driving noise can be recovered by direct observations of the processes by imposing the condition that the Fourier transform of the kernel never vanishes. In particular, we verify that the kernel is total in the Orlicz space induced by the characteristic triplet of the background Lévy process.

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